Overflow Rules and a Weakening of Structural Completeness

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By a **logic** we mean here any system $\mathcal{L} = \langle A, R \rangle$ where A is a set of axioms and R is a set of standard rules determining a structural, finitistic inference relation $\vdash_{\mathcal{L}}$ of the logic $\mathcal{L}$. By $\mathcal{L}$-**tautologies** we mean elements of the set $\{\varphi : \emptyset \vdash_{\mathcal{L}} \varphi\}$. If $\varphi_1, \ldots, \varphi_n \vdash_{\mathcal{L}} \psi$ then a rule of the form:

$$\frac{\varphi_1, \ldots \varphi_n}{\psi}$$

is said to be $\mathcal{L}$-**derivable**. The rule is $\mathcal{L}$-**admissible** if it preserves $\mathcal{L}$-tautologies, i.e. for every substitution σ, if $\sigma(\varphi_1), \ldots, \sigma(\varphi_n)$ are $\mathcal{L}$-tautologies then so is $\sigma(\psi)$. The logic $\mathcal{L}$ is **structurally complete** if all $\mathcal{L}$-admissible rules are $\mathcal{L}$-derivable.

This old concept introduced by W.A. Pogorzelski [1971] keeps attracting the attention of researchers. See, for example, the invited lecture of J. Raftery *Structural completeness in substructural logic* delivered at the conference *Algebraic and Topological Methods in Non-Classical Logic II* (Barcelona, 15–18 June, 2005).

A number of results on structural completeness of propositional calculi appeared in the seventies in Reports on Mathematical Logic and in Studia Logica. Note that the classical propositional logic and all linear intermediate logics are structurally complete while Łukasiewicz propositional logics $Ł_n = \langle A_n, MP \rangle, n = 3, \ldots, \infty$ are not. Similarly, the intuitionistic propositional logic $INT = \langle A_{INT}, MP \rangle$ is not structurally complete but some disjunctionless fragments such as: $INT_{\{\rightarrow\}}$, $INT_{\{\rightarrow,\wedge\}}$, $INT_{\{\rightarrow,\wedge,\neg\}}$ are. An example of a disjunctionless fragment for which structural completeness fails to hold is the fragment $INT_{\{\rightarrow,\neg\}}$.

A nice early result is a simple sufficient condition of structural incompleteness discovered by J. Perzanowski [1972, 1973] and named by him **linguistic gap**. The result of the present paper can be thought of as a kind of refinement of Perzanowski's idea of linguistic gap because the property of **overflow completeness** is closely related to the absence of linguistic gap.

We start off with a reminder of some algebraic trivia. Every class of similar algebras $\mathbb{K}$ gives rise to a system of **equational logic** whose inference relation $\vdash_{\mathbb{K}}$ is determined by all **quasi-identities** of the form:

$$p_1 \approx q_1 \wedge \ldots \wedge p_n \approx q_n \Longrightarrow p \approx q$$

which are true in $\mathbb{K}$. Since **rules** of equational logic are just quasi-identities then the logic of $\mathbb{K}$ is the same as the logic of the quasi-variety $\mathbf{q}\mathbb{K}$ determined by $\mathbb{K}$ ($\mathbf{q}\mathbb{K} = \mathsf{SPP_u}(\mathbb{K})$). Now, a $\mathbb{K}$-**tautology** is simply an identity of the form $p \approx q$ which is true in $\mathbb{K}$ and a quasi-identity $p_1 \approx q_1 \wedge \ldots \wedge p_n \approx q_n \Longrightarrow p \approx q$ is $\mathbb{K}$-**admissible** if it preserves $\mathbb{K}$-tautologies, i.e. for every substitution σ, if all identities: $\sigma(p_1) \approx \sigma(q_1), \ldots, \sigma(p_n) \approx \sigma(q_n)$ are $\mathbb{K}$-tautologies then so is $\sigma(p) \approx \sigma(q)$. It is easy to see that $\mathbb{K}$-admissible quasi-identities are just those which hold in the ω-generated free algebra $\mathfrak{F}_{\mathbb{K}}(\omega)$ of the quasi-variety generated by $\mathbb{K}$.

The following algebraic version of structural completeness was introduced by C. Bergman [1991]: a class $\mathbb{K}$ of similar algebras is **structurally complete** if $\mathbb{K}$-admissible quasi-identities are $\mathbb{K}$-derivable i.e. if quasi-identities holding in the ω-generated free algebra $\mathfrak{F}_{\mathbb{K}}(\omega)$ hold in all algebras of $\mathbb{K}$ or – in other words – if $\mathbf{q}\mathbb{K} = \mathsf{SPP_u}(\mathfrak{F}_{\mathbb{K}}(\omega))$. Note an observation of Bergman [1991], which nicely exhibits a connection of structural completeness with completeness:

Fact 1. *Let $\mathbb{K}^+$ be the class of all non-trivial members of a structurally complete variety of algebras $\mathbb{K}$. Then for every positive existential sentence φ, $\mathsf{Th}(\mathbb{K}^+) \cap \{\varphi, \neg\varphi\} \neq \emptyset$.*

By an **overflow rule** we mean a quasi-identity of the form

$$p_1 \approx q_1 \wedge \ldots \wedge p_n \approx q_n \Longrightarrow x \approx y$$

where x, y are distinct variables not occurring in the antecedent of the implication. We say that a class of algebras $\mathbb{K}$ is **overflow complete** if all $\mathbb{K}$-admissible overflow rules are $\mathbb{K}$-derivable. Thus, overflow completeness is just a weakening of structural completeness obtained by imposing a simple restriction on the form of considered rules. This restriction, however, makes overflow completeness an exact counterpart of the thesis of Fact 1. Indeed, we have:

Fact 2. *Let $\mathbb{K}^+$ be the class of all non-trivial members of a quasi-variety of algebras $\mathbb{K}$. Then the following conditions are equivalent:*

(1) $\mathbb{K}$ *is overflow complete,*

(2) $\mathsf{Th}(\mathbb{K}^+) \cap \{\varphi, \neg\varphi\} \neq \emptyset$, *for every positive existential sentence φ.*

Proof. (2)⇒(1) Without loss of generality we can assume that $\mathbb{K}^+ \neq \emptyset$. Suppose (2) and consider a $\mathbb{K}$-admissible overflow rule: $p_1 \approx q_1 \wedge \ldots \wedge p_n \approx q_n \Longrightarrow x \approx y$. Since the free algebra $\mathfrak{F}_{\mathbb{K}}(\omega)$ is non-trivial, it must verify the sentence $\neg\exists \ldots [p_1 \approx q_1 \wedge \ldots \wedge p_n \approx q_n]$. This implies that $\mathbb{K}^+ \not\models \exists \ldots [p_1 \approx q_1 \wedge \ldots \wedge p_n \approx q_n]$ and the condition (2) yields that $K^+ \models \neg\exists \ldots [p_1 \approx q_1 \wedge \ldots \wedge p_n \approx q_n]$ which means that our overflow rule is $\mathbb{K}$-derivable. We have proved that $\mathbb{K}$-admissible overflow rules are $\mathbb{K}$-derivable.

(1)⇒(2) Suppose (1) and consider a positive existential sentence φ such that $\mathbb{K}^+ \not\models \varphi$. Let $\mathfrak{A} \in \mathbb{K}^+$ be such that $\mathfrak{A} \not\models \varphi$. Without loss of generality we can assume that $\varphi = \exists \ldots [\psi_1 \vee \ldots \vee \psi_n]$ where each ψ_i is a conjunction of identities. From the assumption it follows that $\neg\exists \ldots [\psi_1 \vee \ldots \vee \psi_n]$ must be true in every algebra having $\mathfrak{A}$ among its homomorphic images. Let $\mathfrak{F}$ be a free algebra of $\mathbb{K}$ with an infinite number of free generators and such that $\mathfrak{A} \in \mathsf{H}(\mathfrak{F})$. Since $\mathfrak{F} \models \neg\exists \ldots [\psi_1 \vee \ldots \vee \psi_n]$ then for every $i = 1, \ldots n$, $\mathfrak{F} \models \neg\exists \ldots \psi_i$ and this means that all overflow rules of the form $\psi_i \Longrightarrow x \approx y$, $i = 1, \ldots n$, are true in $\mathfrak{F}$ and thus, $\mathbb{K}$-admissible. Applying (1) we get that $\mathbb{K} \models \forall \ldots [\psi_i \Longrightarrow x \approx y]$, for every $i = 1, \ldots n$ which yields that $\mathbb{K}^+ \models \neg\exists \ldots \psi_1 \wedge \ldots \wedge \neg\exists \ldots \psi_n$ and thus $\mathbb{K}^+ \models \neg\varphi$. Q.E.D.

The fact that overflow completeness could also be called **existential positive completeness** suggests a natural question of characterizing overflow complete quasi-varieties in terms of some algebraic operations applied to free algebras.

Let $\mathbb{H}$ be the variety of Heyting algebras with basic operations: $\vee, \wedge, \rightarrow, \neg$ and let τ be a set of term-operations of Heyting algebras. For every $\mathbb{K} \subseteq \mathbb{H}$, by $\mathbb{K}_\tau$ we shall denote the variety of type τ generated by all τ-reducts of members of $\mathbb{K}$. The following fact is a consequence of the Glivenko theorem:

Fact 3. *If both constants:* $\top$(**verum**) and $\bot$(**falsum**) *are τ-definable and distinct in all non-trivial algebras of the variety $\mathbb{K}_\tau$ then $\mathbb{K}_\tau$ is overflow complete.*

Proof. Take any $\mathbb{K} \subseteq \mathbb{H}$ and a set τ of term operations satisfying the required condition. Suppose that an overflow rule: $p_1 \approx q_1 \wedge \ldots \wedge p_n \approx q_n \Longrightarrow x \approx y$ is not $\mathbb{K}_\tau$-derivable, i.e. $\mathbb{K}_\tau \not\models \forall \ldots [p_1 \approx q_1 \wedge \ldots \wedge p_n \approx q_n \Longrightarrow x \approx y]$. Then there exists $\mathfrak{A} \in \mathbb{K}$ such that $\mathfrak{A} \not\models \forall \ldots [p_1 \approx q_1 \wedge \ldots \wedge p_n \approx q_n \Longrightarrow x \approx y]$ which implies that $\mathfrak{A}$ is non-trivial and $\mathfrak{A} \models \exists \ldots [p_1 \approx q_1 \wedge \ldots \wedge p_n \approx q_n]$. Since $\mathfrak{A} \not\models \top \approx \bot$ then $INT \not\vdash [p_1 \leftrightarrow q_1 \wedge \ldots \wedge p_n \leftrightarrow q_n] \leftrightarrow \bot$, and applying Glivenko theorem, we infer that the formula $\neg[p_1 \leftrightarrow q_1 \wedge \ldots \wedge p_n \leftrightarrow q_n]$ is not a classical tautology. Let v be a (0,1)-valuation falsifying our formula in the two-element Boolean algebra. We define a special substitution σ_v determined by the valuation v putting, for every variable z, $\sigma_v(z) := \bot$ if $v(z) = 0$ and $\sigma_v(z) := \top$ otherwise. It is easy to see that

$\mathbb{H} \models \sigma_v(p_i) \approx \sigma_v(q_i)$, $i = 1, \ldots, n$, which means that the considered overflow rule is not $\mathbb{K}_\tau$-admissible. Q.E.D.

Even though both constants: $\top$(**verum**) and $\bot$(**falsum**) are $\{\leftrightarrow, \neg\}$-definable ($\top := x \leftrightarrow x$, $\bot := \neg\top$), Fact 3 cannot be applied to $\{\leftrightarrow, \neg\}$-reducts because each variety generated by the $\{\leftrightarrow, \neg\}$-reduct of a non-trivial Heyting algebra contains a non-trivial member in which the identity $\top \approx \bot$ is true. In consequence, every non-trivial variety of the form $\mathbb{K}_{\{\leftrightarrow, \neg\}}$, where $\mathbb{K} \subseteq \mathbb{H}$, is not overflow complete because it has an admissible but underivable overflow rule of the form: $\top \approx \bot \Longrightarrow x \approx y$.

Bibliography

C. BERGMAN (1991), Structural completeness in algebra and logic, in: *Algebraic Logic*, H. Andréka, J.D. Monk, and I. Németi (eds.), Colloq. Math. Soc. János Bolyai, vol. 54, North-Holland Publishing Co., Amsterdam, pp. 59–73 (Proc. Conf. Budapest 1988).

J. PERZANOWSKI (1972), A linguistic criterion of structural incompleteness, *Bull. of the Section of Logic, Polish Acad. Sci.*, 1, pp. 18–20.

———— (1973), The linguistic criterion of structural incompleteness, *Reports on Mathematical Logic*, 1, pp. 13–14.

W.A. POGORZELSKI (1971), Structural completeness of the propositional calculus, *Bulletin de l'Académie Polonaise des Sciences, Série des sciences math., astr. et phys.*, XIX, pp. 349–351.